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ON THE RECOVERY OF GRAVITY ANOMALIES FROM HIGH PRECISION ALTIMETER DATA

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Foreword

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Abstract

A model for the recovery of gravity anomalies from high precision altimeter data is derived which consists of small correction terms to the inverse Stokes' formula. The influence of unknown sea surface topography in the case of meandering currents such as the Gulf stream is discussed. A formula was derived in order to estimate the accuracy of the gravity anomalies from the known accuracy of the altimeter data. It is shown that for the case of known harmonic coefficients of lower order the range of integration in Stokes inverse formula can be reduced very much.

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1. Introduction

As part of its Earth and Ocean Physics Application Program (EOPAP) the National Aeronautics and Space Administration (NASA) plans in the next ten years the launch of some satellites equipped with altimeter for ranging to the ocean surface. The announced accuracy of the future altimeter systems lies in the range of 10 cm.

Altimeter measurements of this accuracy will give an important increase in the knowledge of the features of the gravity potential of the earth, because the ocean surface is not far apart from an equipotential surface of the gravity potential. In the present report a model for the recovery of gravity anomalies from this high precision altimeter data will be developed.

In a study of this kind three questions should be answered:

- 1) What is the mathematical model for the solution of the problem, which is reasonably simple but sufficiently accurate in view of the given accuracy of the data?
- 2) Can this theoretical model be transformed into a stable numerical procedure, suited for a modern computer?
- 3) What is the accuracy of the result that can be expected in view of the announced accuracy of the data?

The next section is dedicated to a detailed description of the theoretical model. It is based on a solution of Stokes problem, which was developed in a preparatory study (Lelgemann, 1975). The present procedure uses the formulae, which were derived there, in an inverse application. With this idea the solution consists of the three steps:

- a) Correct the altimeter measurements by some small

terms to a suited quantity N_a ,

$$(1-1) \quad N_a = \frac{T_e}{\gamma_e}$$

γ_e normal gravity at equator

- b) Use the well-known inverse Stokes' formula for a spherical problem

$$(1-2) \quad \Delta g_a = - \left(\frac{\gamma_e}{a} N_a + \frac{\gamma_e}{4\pi a} \int_{\sigma} M(\psi) (N_a - N_{ap}) d\sigma \right)$$

a semi major axis

- c) Correct the result Δg_a by some small terms to the desired gravity anomaly Δg at the earth's surface.

A detailed explanation of the method and the terms in formula (1-1) and (1-2) is given in the next section. The method is well suited to the reduction of our complicated problem to a solution of a spherical problem.

As there are no numerical problems in the computation of the correction terms the answer to question two is mostly concerned with the spherical solution. In the case of spherical approximation much work has already been done in answering this question. A direct and also indirect solution of formula (1-2) is discussed by (Gopalapillai, 1974). (Smith, 1974) and (Rapp, 1974) both applied the method of least squares estimation to the spherical problem.

The main work done in the present study is the consideration of the accuracy, which can be expected under some favourable assumptions about the size of the measuring error and the distribution of the measurements. Under favourable circumstances (a good knowledge of the satellite orbit and so on) there

remain in any case three error sources:

- a) Influence of the unknown sea surface topography. We shall discuss in section three the size and influence of this error type.
- b) The deficiency of suitable altimeter data on the continents. We may overcome this difficulty by neglecting the outer zones in the integral (1-2); that means, we will integrate only up to a certain distance ψ_0 . For this reason we have to discuss the error due to the neglected parts of the potential function itself.
- c) Influence of measuring errors. We will assume that mean values in $b(km) * b(km)$ blocks are given with an accuracy of $m(N)$, e.g. $F_b = b * b = 10km * 10km$ and $m(N) = \pm 10cm$.

The problem under consideration is the behavior of these errors when they are transformed by a linear integral operator over the sphere, especially in view of the fact, that the kernel

$$(1-3) \quad M(\psi) = \frac{2}{\ell_0^3} = \frac{2}{(2 \sin \psi/2)^3}$$

ψ spherical distance

ℓ_0 distance on the unit sphere

of the integral is singular, so that this integral exists only under some restrictions.

Problems of this form are discussed within statistical methods in physical geodesy (Heiskanen-Moritz, 1967). A deep examination of the problem of the error propagation in the case of integral operators over the unit sphere

was given by Meissl (Meissl, 1971). From both these publications we will recall some basic relations and definitions about functions on the sphere, which we need in our special analysis.

A function on the surface of the sphere, which is at least Lebesgue-integrable, can be expanded into a convergent series of spherical harmonics by

$$(1-4) \quad f(\theta, \lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^n [\bar{a}_{nm} \bar{R}_{nm}(\theta, \lambda) + \bar{b}_{nm} \bar{S}_{nm}(\theta, \lambda)]$$

$\bar{R}_{nm}(\theta, \lambda), \bar{S}_{nm}(\theta, \lambda) \dots$ fully normalized spherical harmonics.

The terms

$$(1-5) \quad f_n^2 = \sum_{m=0}^n (\bar{a}_{nm}^2 + \bar{b}_{nm}^2)$$

are called the degree variances of the function f ,

$$(1-6) \quad \text{cov}(f, f) = \sum_{n=2}^{\infty} f_n^2 P_n(\cos \psi)$$

the covariance function of the function f (Heiskanen-Moritz, 1967, (7-19)) and

$$(1-7) \quad \text{var}(f^2) = \|f\|^2 = \sum_{n=2}^{\infty} f_n^2 < \infty$$

the variance (or the norm square) of the function f . Because it gives all information about the statistical behavior of the disturbing potential, the covariance function $\text{cov}(N, N)$ or $\text{cov}(\Delta g, \Delta g)$ is a basic tool of the error considerations in physical geodesy.

At present the best information about the statistical behavior of the dis-

turbing potential on the geoid is probably the covariance function given by Tscherning and Rapp (Tscherning-Rapp, 1974),

$$(1-8) \quad \text{cov}(N, N) = \sum_{n=2}^{\infty} \left(\frac{a^2}{\gamma_0^2} \right) \frac{A}{(n-1)(n-2)(n+B)} s^{(n+1)} P_n(\cos \psi),$$

with

$$\begin{aligned} A &= 425.28 \text{ mgal}^2 \\ B &= 24 \\ S &= 0.999617. \end{aligned}$$

A covariance function of this form presupposed that the function N is analytic on the sphere (that is on the geoid). In the series for the covariance function, this presupposition is expressed by the factor $(s^{(n+1)})$, which runs very fast to zero.

We will try to make some specifications about the covariance function of the error function ν . It is well known that in the case of uncorrelated point measurements the error covariance function has a certain value at the origin and is zero elsewhere. In accordance with our presupposition we have as measurements \bar{N} mean values in certain blocks of area F_b , that is, the function \bar{N} is given as a step function with discontinuities at the boundaries of the blocks. In this case we have to consider not individual measuring errors rather an error function

$$(1-9) \quad \nu = N - \bar{N}.$$

This error function ν must be likewise a bounded step function. Therefore ν is Lebesgue-integrable over the sphere with the series evaluation

$$(1-10) \quad \nu(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^n [\bar{\alpha}_{nm} \bar{R}_{nm}(\theta, \lambda) + \bar{\beta}_{nm} \bar{S}_{nm}(\theta, \lambda)]$$

under the condition

$$(1-11) \quad m^2(N) = \| \nu \|^2 = \sum_{n=2}^{\infty} \nu_n^2 < \infty,$$

where ν_n is defined as

$$(1-12) \quad \nu_n^2 = \sum_{m=0}^n (\overline{\alpha}_{nm}^2 + \overline{B}_{nm}^2).$$

We need both the covariance function of the potential function itself and the error covariance function in our discussion about the error propagation.

Within the estimate of the error propagation in the case of an integral operator on the sphere, a very useful tool is the conception of the greatest lower bound and the least upper bound (Meissl, 1971, p. 26). For example, we can write for the norm between N and Δg

$$(1-13) \quad \text{GLB}(|\lambda_n|) < \frac{\|\Delta g\|}{\|N\|} < \text{LUB}(|\lambda_n|).$$

GLB is the abbreviation of greatest lower bound, some sort of a generalized minimum and LUB stands for least upper bound, which is some sort of generalized maximum. Under very general conditions the GLB and LUB of the norm ratios of the main functions of gravimetric geodesy are computed by Meissl (Meissl, 1971).

In our special problem we have

$$(1-14) \quad N_{nm} = \frac{a}{\gamma_0} \cdot \frac{1}{(n-1)} \Delta g_{nm}.$$

Since we know the covariance function of the disturbing potential itself we can compute the norm ratio explicitly by

$$(1-15) \quad \frac{\gamma_0}{a} < \frac{\|\Delta g\|}{\|N\|} = \sqrt{\frac{\text{var}(\Delta g)}{\text{var}(N)}}.$$

Using the covariance function given by (Tscherning-Rapp, 1974) we get

$$(1-16) \quad \frac{\gamma_0}{a} < \frac{\|\Delta g\|}{\|N\|} \doteq 1.4 \cdot \frac{\gamma_0}{a},$$

because the disturbing potential is approximated by an analytical function.

In the case of the error function we have, corresponding to (1-14), the relationship

$$(1-17) \quad \nu_{nm} = \frac{a}{\gamma_0} \left(\frac{1}{n-1} \right) \epsilon_{nm}$$

ϵ error of the gravity anomalies Δg

with an estimate of the norm ratio of

$$(1-18) \quad \frac{\gamma_0}{a} < \frac{\|\epsilon\|}{\|\nu\|} < \infty$$

(see (Meissl, 1971, page 46)). The difference in the expressions (1-16) and (1-18) is of course due to the fact, that the error function ν is not an analytical function.

We see from (1-18), that small errors in the data may accumulate in unfavorable cases to infinite large errors in the wanted result. There are two

principal possibilities to overcome this undesirable behavior:

- a) Smoothing out also the error function ν to an analytical function. This is possible by a suitable interpolation procedure.
- b) Smoothing out the operator.

According to the assumed form of our measurements as mean values of certain blocks we shall use in our accuracy considerations the second possibility. In geodetic problems this is usually done by a separate treatment of a certain cap with radius ψ_1 around the point under consideration. In our particular case this can be done by subtracting from the measurements N the mean value \bar{N}_p of the central block, that is for the inner zone,

$$(1-19) \quad \bar{N} - \bar{N}_p = 0.$$

The error function of the geoidal undulations in the inner zone is then of course likewise zero. On the other hand the neglected potential function itself will generate an error.

We will give now a summary of the results of the error considerations. All the derivations in the further sections are done only for point values. Under some additional assumptions we will enlarge these results also to the error of mean values. The point error can be separated into three terms;

$$(1-20) \quad m^2(\Delta g) = m^2(\Delta g_n) + m^2(\Delta g_{ss}) + m^2(\Delta g_{oz})$$

$m(\Delta g)$ mean point error

$m(\Delta g_n)$ mean error due to data error and the neglect of the inner zone.

$m(\Delta g_{ss})$ mean error due to unknown or
variable sea surface topography

$m(\Delta g_{oz})$ mean error due to the neglect
of the outer zone.

Under these three types of individual errors, $m(\Delta g_m)$ seems by far the most critical. Therefore, we will consider this error first.

The following formulae and numerical values are dependent on the main assumption that mean geoidal undulations in blocks of size b [km] * b [km] and with an accuracy of $m(N)$ [cm] are given. The number 0.34 in the second term of (1-17) depends on an estimate of the root of the variance of the gravity gradient of

$$\left\| \frac{\partial \Delta g}{\partial r} \right\| = \pm 24 \text{ EU},$$

EU Eötvös unit

drawn from (Metzger and Jircitano, 1974)

Under these assumptions the accuracy of point anomalies at the midpoint of the blocks can be estimated by

$$(1-21) \quad m^2(\Delta g_m) = \frac{6.1^2 m^2(N)}{b^2} + 0.34^2 b^2$$

($m(\Delta g_m)$ in [mgal], $m(N)$ in [cm], and b in [km]).

The first term of the right hand side is due to measuring errors $m(N)$, the second term is a result of the neglected potential function in the inner zone.

With the aid of this formula we will now give an estimate of the accuracy of mean anomalies of block size

$$c > b.$$

In order to do this we make two additional assumptions:

- 1) The error of representation for sufficiently small blocks of size b is small as compared with the point error of the midpoint due to measuring errors. In this way we can consider (1-21) also as an error estimate for the mean anomaly values of small blocks.
- 2) The error values between neighboring blocks are only slightly correlated. If they are not correlated we get as error of a mean anomaly of block length c

$$m_0(\Delta g_n) \doteq \frac{b}{c} m(\Delta g_n),$$

if they are very strong correlated we get

$$m_0(\Delta g_n) \doteq m(\Delta g_n).$$

As an estimate of the error of mean anomaly values of block size c we obtain therefore

$$(1-22) \quad \frac{b}{c} m(\Delta g_n) \leq m_0(\Delta g_n) < m(\Delta g_n).$$

The correlation between errors of neighboring blocks depends on the

correlation of the gravity gradient in the midpoints and on the correlation due to the measuring error. It is known that the covariance function of the gravity gradients goes very fast to zero (Moritz, 1974). According to the results of Gopalapillai (Gopalapillai, 1974, p.77), the error covariances between $1^\circ \times 1^\circ$ blocks are also very small. So it seems not unlikely that the upper error limit in (1-22) can be reduced further.

The following two examples give an idea about the accuracy of the mean anomalies. In order to get a satisfactory ratio between the two error sources (that is, due to measuring errors and due to the neglected inner zone) we assume for both terms on the right hand side of (1-17) the same magnitude. This is attainable by a block length of

$$(1-23) \quad b = \sqrt{\frac{6.1 \cdot m(N)}{0.34}} .$$

With

$$m(N) = \pm 100 \text{ cm}, \quad b \doteq 42 \text{ km}$$

we get according to (1-22)

$$c = 1^\circ \times 1^\circ : \quad \pm 8 \text{ mgal} < m_c(\Delta g_a) < \pm 20 \text{ mgal}$$

$$c = 2^\circ \times 2^\circ : \quad \pm 4 \text{ mgal} < m_c(\Delta g_a) < \pm 20 \text{ mgal}.$$

In the case of

$$m(N) = \pm 10 \text{ cm}, \quad b \doteq 13 \text{ km}$$

we get

$$c = 1^\circ \times 1^\circ : \pm 1 \text{ mgal} \leq m_0(\Delta g_0) \leq \pm 6 \text{ mgal}.$$

These results are in good agreement with the results of the error considerations in (Rapp, 1974).

We will conclude this introduction with a few comments on the two other error sources. In section three it is assumed that the sea surface topography is known with the exception of meandering currents such as the gulf stream. An estimate of the error due to meandering currents is very difficult. A very rough estimate gave

$$\pm 3 \text{ mgal} \leq m(\Delta g_{ss}) \leq \pm 7 \text{ mgal}.$$

These figures are point errors. However a systematic behavior, which would result in errors of the mean anomalies of the same magnitude, cannot be excluded. Hence attention is required in areas of strong boundary currents such as the gulf stream.

If the gravity field of lower order is known the range of integration can be drastically reduced without remarkable additional error. In this case it is possible to compute likewise gravity anomalies near to the coast. However, the remaining small errors are very strongly correlated at neighboring blocks.

The error for several different circles ψ_0 are given for a gravity field known up to $n_0 = 2$, $n_0 = 8$ and $n_0 = 16$ in section 6. In any case the influence of the constant term N_p in the neglected part

$$\Delta g = - \frac{\gamma_0}{4\pi a} \int_{\psi=\psi_0}^{\pi} \int_{\alpha=0}^{2\pi} M(\psi)(N - N_p) \sin\psi \, d\psi \, d\alpha$$

may become fairly large and must be taken into account in some way.

The results given above are of course only a first estimate. They cannot replace test computations with true altimeter data and perhaps a comparison with accurate gravity anomalies, but they may give a good insight into the difficulties and the limits of the solution.

2. On the Solution of the Inverse Stokes' Problem in Precise Gravimetric Geodesy

The content of this section is entirely based on the results in (Lelgemann, 1975). In that study the gravity anomalies at the earth's surface were reduced to gravity anomalies at the sphere with radius a . Then, the disturbing potential at this sphere was computed with Stokes' formula, combining the result with satellite derived information about the gravity field. From the disturbing potential at this sphere the disturbing potential on the earth's surface was computed with the quasi-geoidal undulations, which can be found by the use of Bruns equation

$$(2-1) \quad \zeta = \frac{T}{\gamma}$$

ζ quasi-geoidal undulation
 γ normal gravity

(In the present section we have to distinguish between the quasi-geoidal undulation ζ and the quantity N_s , defined by (2-19). The geoidal undulation N , used in all other sections, can then be identified with N_s or can be considered as ζ in spherical approximation).

In this report we are going precisely the reverse way. We transform quasi-geoidal undulations (as a result of altimeter measurements) into the disturbing potential, reduce this disturbing potential to the disturbing potential at the sphere a , solve the inverse problem of Stokes for the sphere and reduce the computed gravity anomalies to gravity anomalies at the surface of the earth.

First we shall compile some well-known integral formulae. All these formulae are valid if the range of integration is a sphere. We consider a spherical polar coordinate system with r (radius vector), θ (polar distance) and λ (longitude) and a fixed sphere with $r = a = \text{constant}$. On this sphere a continuous function x should be given. An evaluation of x in a series of spherical harmonics should not contain terms of zero and first order. Now we can define a second function y on the sphere by

$$(2-2) \quad y = \frac{a}{4\pi} \int_0 x S(\psi) d\sigma.$$

This is Stokes' well-known formula applied to a fictitious function x . The function $S(\psi)$ is given by

$$(2-3) \quad S(\psi) = \frac{1}{\sin(\psi/2)} - 6 \sin \psi/2 + 1 - 5 \cos \psi - 3 \cos \psi \ln(\sin \psi/2 + \sin^2 \psi/2)$$

or as a series of spherical harmonics

$$(2-4) \quad S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(\cos \psi).$$

We want a formula for the inverse problem, i. e. we will compute for a given function y a function x . For this we can consider the integral formula (2-1) as a Fredholm integral equation of the first kind. In a general case, the solution may become a very complicated problem, but in the case of the sphere we get the solution easily by a development of the functions involved into a series of eigenfunctions, that is, of spherical harmonics. Due to the orthogonality relations of spherical harmonics we have

$$(2-5) \quad \sum_{n=2}^{\infty} y_n = a \sum_{n=2}^{\infty} \frac{1}{n-1} x_n$$

or by a comparison of the same terms of both sides of the equation

$$(2-6) \quad a x_n = (n-1) y_n.$$

For the similar series

$$(2-7) \quad \sum_{n=2}^{\infty} z_n = \sum_{n=2}^{\infty} n \cdot y_n$$

or

$$(2-8) \quad z_n = n \cdot y_n$$

a closed solution is known by the integral (Heiskanen-Moritz, 1967, p. 39)

$$(2-9) \quad z = - \frac{a^3}{2\pi} \int_G \frac{y - y_p}{\ell^3} d\sigma$$

with

$$\ell = 2 a \sin \psi/2,$$

provided y is differentiable at least twice at the point P under consideration.

We will write (2-9) in the following form

$$(2-10) \quad z = - \frac{1}{4\pi} \int_G M(\psi) (y - y_p) d\sigma$$

with

$$(2-11) \quad M(\psi) = \frac{2}{(2 \sin \psi/2)^3}.$$

Writing equation (2-6) in the following form

$$(ax + y)_n = ny_n$$

we obtain

$$ax + y = - \frac{1}{4\pi} \int_{\sigma} M(\psi) (y - y_p) d\sigma$$

or

$$(2-12) \quad x = - \frac{y}{a} - \frac{1}{4\pi a} \int_{\sigma} M(\psi) (y - y_p) d\sigma$$

or in series form

$$(2-13) \quad x = \frac{1}{a} \sum_{n=1}^{\infty} (n-1) y_n.$$

Assuming in this section that the results of altimetry are quasi-geoidal undulations ζ we neglect for the moment the influence of sea surface topography. We shall return to the more realistic case in the next section considering then the sea surface topography as an error source.

As mentioned above the computation of Δg values can be managed in precisely the reverse way than the computation of geoidal undulations in (Lelgemann, 1975, section 4). We will use here the same notation and abbreviations. We solve the problem likewise in eight successive steps:

- 1) Transformation from geoidal undulations ζ into the disturbing potential T at the surface of the ocean by

$$(2-14) \quad T = \zeta \cdot \gamma$$

γ normal gravity

Formulae for γ are given in (IAG, 1970, p. 58)

- 2) The correction due to the indirect effect of atmospheric gravity reduction was computed to be maximal 0.6 cm. So we can set

$$(2-15) \quad \delta t_1 = 0.$$

- 3) Correction due to the upward continuation term because of topography

$$(2-16) \quad T_E = T - \delta t_2$$

$$\delta t_2 = -H \cdot \Delta g$$

H orthometric height
 Δg gravity anomaly

This correction is necessary in the neighborhood of coasts and on islands. On the sea we have, of course, due to $H=0$

$$\delta t_2 = 0.$$

- 4) Correction because of the ellipticity of the reference surface. The computation of the disturbing potential at the sphere with radius a is found from

$$(2-17) \quad T_a = T_E - \delta t_3$$

where

$$(2-18) \quad \delta t_3 = \frac{e'^2}{4} \cos^2 \theta \cdot T(\theta, \lambda)$$

5) We write T_a in the following form

$$(2-19) \quad T_a = N_a \cdot \gamma_0.$$

γ_0 normal gravity at equator

In order to avoid the definition of a geoidal undulation in space, we will consider N_a as an auxiliary quantity defined by (2-19). However, N_a can be considered as the quasi-geoidal undulation in spherical approximation and formula (2-20) may be more familiar than a similar formula, using the potential T_a itself instead of N_a ,

$$(2-20) \quad \Delta g_a = - \left(\frac{\gamma_0}{a} N_a + \frac{\gamma_0}{4\pi a} \int_{\sigma} M(\psi) (N_a - (N_a)_p) d\sigma \right).$$

This is formula (2-12), applied to our special problem.

6) Computation of the gravity anomalies at the ellipsoid from the gravity anomalies on the sphere with radius a :

$$(2-21) \quad \Delta g_E = \Delta g_a + \delta g_3.$$

From (Lelgemann, 1975) we find

$$(2-22) \quad \delta g_3 = - \frac{e'^2}{a} \sum_{n=2}^{\infty} \sum_{m=0}^n (n-1) [C_{nm} R_{nm}(\theta, \lambda) + D_{nm} S_{nm}(\theta, \lambda)]$$

with

$R_{nm}(\theta, \lambda)$, $S_{nm}(\theta, \lambda)$ unnormalized spherical harmonics.
They have the same definition as in
(Heiskanen-Moritz, 1967)

The coefficients C_{nm} and D_{nm} can be computed in the following way. When

$$(2-23) \quad T(\theta, \lambda) = \sum_{n=2}^{\infty} \sum_{m=0}^n [A_{nm} R_{nm}(\theta, \lambda) + B_{nm} S_{nm}(\theta, \lambda)],$$

then

$$(2-24) \quad C_{nm} = A_{(n-2)m} p_{nm} + A_{nm} q_{nm} + A_{(n+2)m} r_{nm}$$

$$D_{nm} = B_{(n-2)m} p_{nm} + B_{nm} q_{nm} + B_{(n+2)m} r_{nm}$$

where

$$(2-25) \quad p_{nm} = \frac{(5n-17)(n-m-1)(n-m)}{4(n-1)(2n-3)(2n-1)}$$

$$q_{nm} = \frac{-6n^3 - 8n^2 + 25n + 6nm^2 + 6m^2 + 21}{4(n-1)(2n+3)(2n-1)}$$

$$r_{nm} = \frac{(5n+11)(n+m+2)(n+m+1)}{4(n-1)(2n+5)(2n+3)}$$

- 7) The gravity correction due to topography is zero, because we want gravity on the sea surface. We have

$$(2-26) \quad \Delta g_s = \Delta g_t + \delta g_2$$

with

$$\delta g_2 = H_p \cdot L_1 (\Delta g),$$

where $L_1 (\Delta g)$ is an abbreviation for

$$L_1 (\Delta g) = \frac{R^2}{2\pi} \iint_0^\pi \frac{\Delta g - \Delta g_p}{\ell_0^3} d\sigma.$$

This correction is only different from zero, if we want to compute gravity anomalies at land. At sea, H_p is zero and therefore we get always

$$\delta g_2 = 0.$$

8) Add to this value the gravity effect δg_1 of the atmosphere

$$(2-27) \quad \Delta g = \Delta g_s + \delta g_1.$$

Because this correction is constant at mean sea level we have

$$\delta g_1 = -0.87 \text{ mgal}.$$

However the recommendation at the IAG (IAG, 1970) may be notified, that this correction may be handled as a measurement correction. In this case we

have of course

$$\delta g_1 = 0.$$

There are no difficulties in the computation of these correction terms. So we can restrict our accuracy considerations to the case of a spherical problem. Only the problem of the unknown sea surface topography needs a special consideration.

3. An Estimate of the Error Influence of the Sea Surface Topography

As Moritz (1974a) has shown we can compute the sea surface topography with geodetic measurements by a combination of altimeter data and gravity observations. However, if we want to compute gravity from the altimeter observations the difference between the real ocean surface and an equipotential surface of the earth's gravity potential must be known.

The difference between the instantaneous sea surface and an equipotential surface of the gravity field is generated by many different forces. Therefore, the computation of this quantity may be very complicated (Wemelsfelder, 1970). For our present purposes it seems sufficient to consider the instantaneous sea surface as an equipotential surface disturbed by irregularities of four types:

- 1) very short periodic irregularities (e.g. ocean waves, swell)
- 2) periodic or quasi-periodic irregularities (e.g. tides)
- 3) quasistationery irregularities, which retain their form but change their places (e.g. due to the meandering Gulf stream)
- 4) quasistationery irregularities, which retain form and place (e.g. due to equatorial currents).

From a geodetic point of view we consider sea surface topography only as an error source. So we will make two suppositions in view of the following error considerations:

- a) The very short periodic irregularities and the periodic and quasi-periodic effects by tides and seasonal effects are corrected either by statistical methods or by suitable models of the disturbing forces.

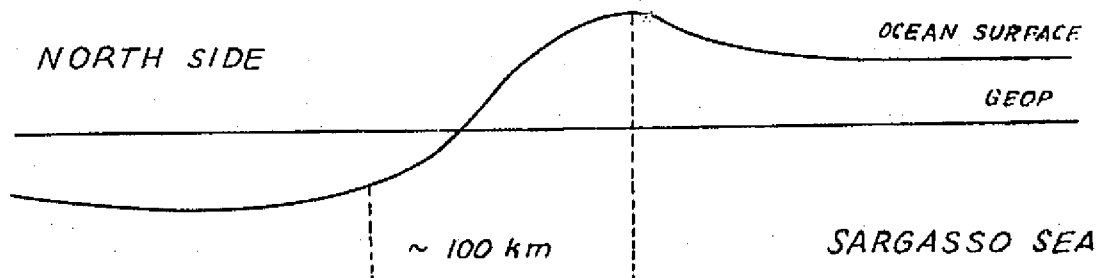
- b) Irregularities of type three and four must be given by oceanographic science or we have to consider these irregularities as a possible error source.

Because of the second supposition, the error discussion was made under the thesis: If oceanography does not need the help of altimetry for a solution of their problems the altimeter data can be corrected to geoidal undulations. If oceanographic sciences want altimeter data for a computation of certain phenomena we will consider these as an error source in our model.

From this point of view the difficulties seem not so important with irregularities of type four, which result from the shape and the magnitude of the large scale features of the mean sea surface. Information is already given in form of maps, from which we will mention the map from Stommel (Sturges, 1972) and the map from Lisitzin (Rapp, 1974). The difference of about 1.5 m between the values in the two maps is largely constant. An estimate of ± 30 cm of the uncertainty of these maps is given by Sturges. He mentions that this error could probably be reduced to approximately ± 10 cm. However, the difference between oceanographic and geodetic results in the large scale features of mean sea level along the coast lines are not yet clarified.

The situation seems not so well defined in the case of strong boundary currents such as the gulf stream and maybe in the case of eddies. The tracking and also the detecting of strong boundary currents and of eddies could be one of the tasks which oceanographic science expects from altimetry (Sturges, 1972).

Strong boundary currents such as the Gulf stream have the following form (Stommel, 1965).



The current flows perpendicular to the plane of the page. The height difference between the warm water of the sargasso sea and the colder water at the north side of the stream is expected to be up to two meters. So we have to expect a slope of the stream of

$$2 \text{ m} : 100 \text{ km} = 2 \cdot 10^{-5} .$$

The main problem is the meandering of the gulf stream. Although the meanders can remain in a fixed position for a short time, they usually change position substantially during a time scale of perhaps two weeks. An expression of this meandering is seen in figure three in (Sturges, 1972).

It seems very difficult to obtain an exact knowledge of this meandering, which is expected in the eastern part of the gulf stream to an horizontal extent of 500 kilometers. So we have to consider the sea surface topography generated from such a current as an data error f_N .

From formula (1-2) we get

$$\begin{aligned} (\Delta g + f_g) = & - \left(\frac{\gamma_0}{a} (N + f_N) + \frac{\gamma_0}{4\pi a} \int_0^\sigma M(\psi) [(N + f_N) \right. \\ & \left. - (N_p + (f_N)_p)] d\sigma \right) \end{aligned}$$

or

$$(3-1) \quad f_g = - \frac{\gamma_0}{a} f_N - \frac{\gamma_0}{4\pi a} \int_0^\sigma M(\psi) (f_N - (f_N)_p) d\sigma .$$

An error estimate based on this formula seems not easy. We will give a rough estimate of this error using in a completely different way our knowledge about the slope of the sea surface topography.

For the ratio between the norm of the slope Δv of the geoid and the norm of the gravity anomalies Δg we draw from (Meissl, 1971, page 42),

$$(3-2) \quad \frac{\gamma_0}{\sqrt{6}} \leq \frac{\|\Delta g\|}{\|\Delta v\|} \leq \gamma_0,$$

where we have let $G = \gamma_0$.

We will apply this formula to the error f_N . We do not know the variance of f_v , but only a maximal value of $\max(f_v) \doteq 2 \cdot 10^{-5}$. We interpret the error due to the sea surface topography of the boundary currents such as the gulf stream as an stochastic process with a normal distribution on the sphere. In this case we have

$$(3-3) \quad \max |f_g| \doteq 3 \cdot \|f_g\| = 3 \cdot m(\Delta g_{ss})$$

and

$$(3-4) \quad \max |f_v| \doteq 3 \cdot \|f_v\| = 3 \cdot m(\Delta v_{ss}).$$

Then we get the following inequality

$$\frac{\gamma_0}{\sqrt{6}} \max |f_v| \leq \max |f_g| \leq \gamma_0 \max |f_v|.$$

With

$$\max |f_v| \doteq 2 \cdot 10^{-5}$$

and

$$\gamma_0 \doteq 980000 \text{ mgal}$$

we get as an rough estimate

$$8 \text{ mgal} \leq \max |f_s| \leq 20 \text{ mgal}$$

or

$$(3-5) \quad \pm 3 \text{ mgal} \leq m(\Delta g_{ss}) \leq \pm 7 \text{ mgal}.$$

Because of the assumptions above the derivation of this estimate is very weak, but it gives at least an expression of the magnitude of this error. A recovery of gravity data in areas of strong boundary currents should be done with uttermost caution, including a very carefully error analysis of data and results.

4. The Error Influence of the Neglected Inner Zone

We have assumed already that our data are mean values of geoidal undulations \bar{N} of blocks of size $F_b = b * b [\text{km}^2]$ with an root mean square error of $m(N)$. We will compute the value Δg_p of the gravity anomaly always at the mid-point P of the blocks. For the value $(N - N_p)$ inside of the inner most block we get

$$(4-1) \quad (\bar{N} - \bar{N}_p) = (\bar{N}_p - \bar{N}_p) = 0,$$

replacing in the integration procedure the function N by its mean value \bar{N} .

We will now estimate the error $m(\Delta g_1)$ generated by this procedure. In view of the fact that we make only an error consideration, the integration over the inner square block may be approximated by an integration over an circular cap of the same area with sufficient accuracy. Because of

$$(4-2) \quad F_b = b * b = s_1^2 \pi$$

we get

$$(4-3) \quad s_1 = \frac{1}{\sqrt{\pi}} \cdot b$$

as a value of the radius s_1 of this circular cap. Approximating the radius ℓ_1 of a spherical cap by the radius s_1 of a plane circle

$$(4-4) \quad \ell_1 = 2a \sin \psi_1 / 2 \doteq a \sin \psi_1 \doteq s_1$$

and the spherical surface element by

$$(4-5) \quad a^2 d\sigma = a^2 \sin\psi d\psi d\alpha = s ds d\alpha,$$

we get for the influence of the inner zone

$$(4-6) \quad \Delta g_1 = - \frac{\gamma_0}{4\pi a} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\psi_1} M(\psi) (N - N_p) \sin\psi d\psi d\alpha$$

$$\doteq - \frac{\gamma_0}{2\pi} \int_{\alpha=0}^{2\pi} \int_{s=0}^{s_1} (N - N_p) \frac{1}{s^2} ds d\alpha.$$

We develop the geoidal undulation N at the point P into a Taylor series

$$(4-7) \quad N = N_p + x N_x + y N_y + \frac{1}{2!} (x^2 N_{xx} + 2xy N_{xy} + y^2 N_{yy}) + \dots$$

with

$$N_x = \left(\frac{\partial N}{\partial x}\right)_p \quad \text{and} \quad N_{xx} = \left(\frac{\partial^2 N}{\partial x^2}\right)_p.$$

Introducing polar coordinates,

$$x = s \cdot \cos\alpha \quad \text{and} \quad y = s \cdot \sin\alpha$$

we get the following expression

$$N = N_p + s(N_x \cos\alpha + N_y \sin\alpha) + \left(\frac{s^2}{2}\right)(N_{xx} \cos^2\alpha$$

$$+ 2N_{xy} \cos\alpha \sin\alpha + N_{yy} \sin^2\alpha).$$

The integration over the azimuth α gives

$$\Delta g_1 = - \frac{\gamma_0}{4} \int_{s=0}^{s_1} (N_{xx} + N_{yy}) ds$$

and the integration over the distance s

$$\Delta g_1 = - \frac{\gamma_0 \cdot s_1}{4} (N_{xx} + N_{yy})$$

or

$$(4-8) \quad \Delta g_1 = - \frac{\gamma_0 \cdot s_1}{4} \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} \right)_p.$$

ξ, η deflections of the vertical

We cannot estimate the error influence from this formula, because we have no statistical information about the size of the horizontal gradient of the deflections of the vertical. However, based on Laplace's equation we can exchange this expression into

$$(4-9) \quad \frac{\partial \Delta g}{\partial r} = -G \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} \right)$$

(Heiskanen-Moritz, 1967, (2-221)).

Because

$$G \doteq \gamma_0$$

we obtain for the influence of the inner zone:

$$(4-10) \quad \Delta g_1 = \frac{s_1}{4} \left(\frac{\partial \Delta g}{\partial r} \right)_p.$$

This expression is better suited for an estimate of the error which appears with the neglect of the inner zone. The variance of the gravity gradient is known to be (Metzger and Jircitano, 1974):

$$\left\| \frac{\partial \Delta g}{\partial r} \right\| = \pm 24 \text{ EU.}$$

EU Eötvös unit

With the relation

$$1 \text{ Eötvös unit} = 1 \text{ EU} = \frac{1 \text{ mgal}}{10 \text{ km}} = 10^{-8} \text{ sec}^{-2}$$

we get as an estimate of the root mean square error of the point anomaly for the midpoint of the inner zone due to the neglecting of this zone

$$m(\Delta g_1) = \pm \frac{24}{10} \cdot \frac{1}{4\sqrt{\pi}} \cdot b \text{ [mgal]}$$

or

$$m(\Delta g_1) = \pm 0.34 \cdot b \text{ [mgal]},$$

where b must be inserted in the dimension km. It is easy to see that the value

0.34 depends very strongly on the estimate of the variance of the gravity gradient of ± 24 EU.

It may be pointed out that the formula (4-10) is valid only for a sufficiently small zone, dependent on the desired accuracy. For a more detailed examination see (Meissl, 1971a).

5. Error Considerations for a Given Accuracy of the Altimeter Data

As in the last section we assume that our data are mean values of geoidal undulations \bar{N} of blocks of the size $F_b = b * b$ [km]² with a root mean square error of $m(N)$. Furthermore we make the assumption that these measurements are homogeneous and uniformly distributed over the whole sphere. The more realistic case, that the mean values of the geoidal undulations are given only in a certain neighborhood of the point under consideration, is discussed in the next section.

We have to study the error propagation in the inverse Stokes' formula

$$(5-1) \quad \Delta g = - \frac{\gamma_0}{4\pi a} \int_0 \left(\bar{N} - \bar{N}_p \right) M(\psi) d\sigma.$$

For the moment we shall consider only the integral part of formula (1-2), taking the linear term into account afterwards.

The main difference as compared with Stokes' formula is the stronger singularity of the kernel $M(\psi)$ at the point of interest. But we have already excluded a neighborhood around this point and the error estimated in the last section. So we can apply for the error considerations the same methods as for the error considerations in the case of the computation of geoidal undulations from gravity anomalies (Heiskanen-Moritz, 1967). The individual error ϵ of Δg is

$$(5-2) \quad \epsilon = - \frac{\gamma_0}{4\pi a} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \nu(\psi, \alpha) M(\psi) \sin\psi d\psi d\alpha$$

where $\nu(\psi, \alpha)$ is the individual error of the geoidal undulations.

We consider only the error function ν , taking the constant term ν_p into account with

$$\frac{\gamma_e \cdot \nu_p}{4 \pi a} \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} M(\psi) \sin \psi \, d\psi \, d\alpha = \frac{\gamma_e \cdot \nu_p}{a}$$

(See Heiskanen-Moritz, 1967, p. 37). Errors of this size can be neglected as discussed at the end of the section.

The square of (5-2) becomes

$$\begin{aligned} \epsilon^2 &= \left(\frac{\gamma_e}{4 \pi a} \right)^2 \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \nu(\psi, \alpha) M(\psi) \sin \psi \, d\psi \, d\alpha \\ &\cdot \int_{\alpha'=0}^{2\pi} \int_{\psi'=0}^{\pi} \nu(\psi', \alpha') M(\psi') \sin \psi' \, d\psi' \, d\alpha'. \end{aligned}$$

Because products of definite integrals may be written as one multiple integral we get

$$\begin{aligned} \epsilon^2 &= \left(\frac{\gamma_e}{4 \pi a} \right)^2 \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \int_{\alpha'=0}^{2\pi} \int_{\psi'=0}^{\pi} \nu(\psi, \alpha) \nu(\psi', \alpha') \\ &\cdot M(\psi) M(\psi') \sin \psi \sin \psi' \, d\psi \, d\alpha \, d\psi' \, d\alpha'. \end{aligned}$$

We form now the average of both sides of this equation arriving at

$$\begin{aligned} (5-3) \quad m^2(\Delta g) &= \left(\frac{\gamma_e}{4 \pi a} \right)^2 \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \int_{\alpha'=0}^{2\pi} \int_{\psi'=0}^{\pi} \sigma(\psi, \alpha, \psi', \alpha') \\ &\cdot M(\psi) M(\psi') \sin \psi \sin \psi' \, d\psi \, d\alpha \, d\psi' \, d\alpha'. \end{aligned}$$

In this formula, $m^2(\Delta g)$ is the error variance of the computed Δg -values and

$$(5-4) \quad \sigma(\psi, \alpha, \psi', \alpha') = \sigma(\theta, \lambda, \theta', \lambda') = \{ \nu(\theta, \lambda) \nu(\theta', \lambda') \}$$

(the braces mean the average over the whole world)

is the error covariance function of the geoidal undulations. This is the general form of the error propagation in Stokes' inverse formula. Of course this formula is too complicated for our purpose to get an estimate of the order of the error. So we include two assumptions which can be drawn from the presuppositions about the measuring data:

- 1) Only errors at neighboring points are correlated.
We can make this assumption if errors of different blocks are uncorrelated. The error covariance function of the geoidal undulations may then assumed to be zero for points further than $b \cdot \sqrt{2}$ apart.
- 2) The accuracy is the same for every point of the earth surface. If we work exclusively with altimeter observations this assumption is only valid at the oceans. For this reason the outer zones must be excluded from the computations. The additional error due to this exclusion will be considered in the next section.

By assumption one the integrant is noticeably different from zero only if $(\theta' \doteq \theta)$ and $(\lambda' \doteq \lambda)$, because the error covariance function may be considered as zero for two points more than $b\sqrt{2}$ apart. Hence, we may approximate

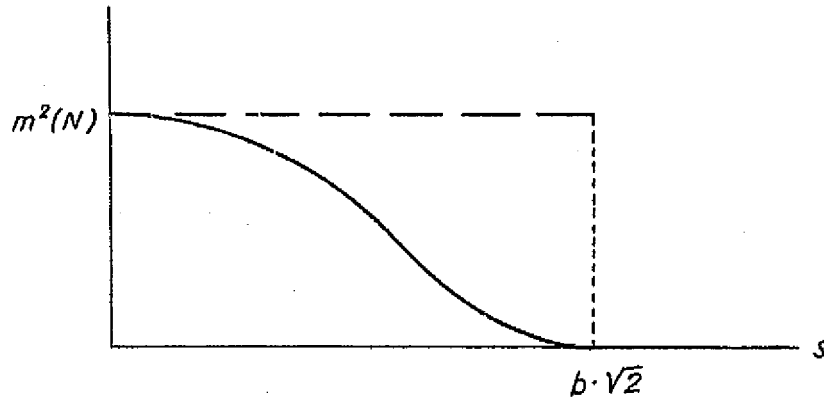
$$(5-5) \quad m^2(\Delta g) = \left(\frac{\gamma_e}{4\pi a} \right)^2 \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \int_{\alpha'=0}^{2\pi} \int_{\psi'=0}^{\pi} \sigma(\psi, \alpha, \psi', \alpha') \\ \cdot M^2(\psi) \sin\psi \sin\psi' d\psi d\alpha d\psi' d\alpha'$$

and perform then the integration over ψ' and α' first. We define

$$(5-6) \quad a^2 \int_{\alpha'=0}^{2\pi} \int_{\psi'=0}^{\pi} \sigma(\psi, \alpha, \psi', \alpha') \sin \psi' d\psi' d\alpha' = E(\psi, \alpha)$$

E error constant of the altimeter data.

As a maximal limit of the real error covariance function, we will assume that the error covariance function is $\text{const} = m(N)^2$ inside of each block and zero outside.



We get under this assumption

$$(5-7) \quad E \doteq m^2(N) \cdot F_b$$

as a maximal estimate of the error constant. Because of assumption two the error constant E is independent of position. In this way we obtain

$$(5-8) \quad m^2(\Delta g) = \left(\frac{\gamma_0}{4\pi a^2} \right)^2 E \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} [M(\psi)]^2 \sin \psi d\psi d\alpha.$$

First we perform the integration with respect to α . We get

$$m^2(\Delta g) = \left(\frac{\gamma_0}{a^2} \right)^2 \frac{E}{8\pi} \int_{\psi=0}^{\pi} [M(\psi)]^2 \sin \psi d\psi.$$

We have already taken into account the small neighborhood $\psi < \psi_1$ in another

way. Consequently, we exclude the origin by beginning the integration with $\psi = \psi_1$

$$m^2(\Delta g) = \left(\frac{\gamma_0}{a^2} \right)^2 \frac{E}{8\pi} \int_{\psi=\psi_1}^{\pi} [M(\psi)]^2 \sin\psi \, d\psi$$

or

$$(5-9) \quad m^2(\Delta g) = \frac{E}{8\pi} \left(\frac{\gamma_0}{a^2} \right)^2 \cdot J_1(\psi_1)$$

with

$$(5-10) \quad J_1(\psi_1) = \int_{\psi=\psi_1}^{\pi} [M(\psi)]^2 \sin\psi \, d\psi.$$

Because of

$$M(\psi) = \frac{2}{2^3 \sin^3 \psi/2}$$

we get

$$J_1(\psi_1) = \frac{1}{16} \int_{\psi=\psi_1}^{\pi} \frac{1}{\sin^6 \psi/2} \sin\psi \, d\psi.$$

In order to solve the integral we introduce the new variable

$$z = \sin\psi/2$$

with

$$\sin \psi_1/2 = t$$

and

$$\sin \psi \, d\psi = 4z \, dz.$$

We can easily verify

$$(5-11) \quad J_1(\psi_1) = \frac{1}{16} \left[\frac{1}{\sin^4 \psi_1/2} - 1 \right].$$

In this way we get as a final result

$$(5-12) \quad m^2(\Delta g) = \frac{E}{128\pi} \left(\frac{\gamma_0}{a^2} \right)^2 \left[\frac{1}{\sin^4 \psi_1/2} - 1 \right].$$

We will now specialize this formula for small angles ψ_1 . We get

$$(5-13) \quad m^2(\Delta g) \doteq \frac{E}{128\pi} \left(\frac{\gamma_0}{a^2} \right)^2 \frac{1}{\sin^4 \psi_1/2}.$$

Taking into account (4-3) we can approximate

$$\sin \psi_1/2 \doteq \psi_1/2 = s_1/2a \doteq b/2\sqrt{\pi}$$

or

$$\frac{1}{\sin^4 \psi_1/2} \doteq \frac{16 \pi^2 a^4}{b^4}.$$

Furthermore, we have from (5-7)

$$E \doteq m^2(N) \cdot b^2.$$

Putting these expressions into (7-13) we arrive with

$$m^2(\Delta g) \doteq \frac{\pi}{8} \frac{\gamma_e^2}{b^2} \cdot m^2(N)$$

or

$$(5-14) \quad m(\Delta g) \doteq \sqrt{\frac{\pi}{8}} \frac{\gamma_e}{b} m(N).$$

This can be written in a form

$$(5-15) \quad m(\Delta g) \doteq \frac{6.1}{b} \cdot m(N)$$

(b in [km], m(N) in [cm], m(Δg) in [mga!]).

Taking finally also the error due to the neglecting of the inner zone into account we get

$$(5-16) \quad m^2(\Delta g_{\Sigma}) = m^2(\Delta g) + m^2(\Delta g_i)$$

or

$$(5-17) \quad m^2(\Delta g_{\Sigma}) = \frac{6.1^2 \cdot m^2(N)}{b^2} + 0.34^2 b^2$$

(b in [km], m(N) in [cm], m(Δg_m) in [mgal]).

This is the variance of the error of the gravity anomaly at the midpoint of a compartment.

We have neglected up to now the influence of the constant term in the inverse Stokes' formula (1-2)

$$(5-18) \quad \frac{\gamma_0}{a} N \doteq 0.15 N \text{ [mgal]}.$$

N has the dimension meter. It is easily seen that an error of 1 m in the geoidal undulation generates an error of 0.15 mgal in the result. This error can certainly be neglected in the error considerations.

6. The Error Influence of Neglected Outer Zones

In the last section it was pointed out that the derived error formula is only valid in the case of geoidal undulations given homogeneously over the whole earth. From altimetry alone there is of course a lack of data over the continents. In the present section we will estimate the error which occurs if we extend the integral only over a certain area, a cup of radius ψ_0 . In this case we neglect the influence of the function itself outside of this cup.

First we split up the integral into

$$(6-1) \quad \Delta g = \Delta g_1 + \Delta g(N_p) + \Delta g_{oz}$$

with

$$(6-2) \quad \Delta g_1 = - \frac{\gamma_e}{4 \pi a} \int_{\psi=0}^{\psi_0} \int_{\alpha=0}^{2\pi} M(\psi) (N - N_p) \sin \psi \, d\psi \, d\alpha$$

$$(6-3) \quad \Delta g(N_p) = N_p \cdot \frac{\gamma_e}{4 \pi a} \int_{\psi=\psi_0}^{\pi} \int_{\alpha=0}^{2\pi} M(\psi) \sin \psi \, d\psi \, d\alpha$$

$$(6-4) \quad \Delta g_{oz} = - \frac{\gamma_e}{4 \pi a} \int_{\psi=\psi_0}^{\pi} \int_{\alpha=0}^{2\pi} M(\psi) \cdot N \cdot \sin \psi \, d\psi \, d\alpha.$$

If we integrate only over the cup $\psi < \psi_0$ we neglect of course the terms $\Delta g(N_p)$ and Δg_{oz} . First we will investigate the magnitude of $\Delta g(N_p)$. Starting the integration in formula (6-3) with the integration over α and inserting $M(\psi)$ by the expression (1-3) we get

$$\Delta g(N_p) = N_p \cdot \frac{\gamma_e}{8 a} \int_{\psi=\psi_0}^{\pi} \frac{1}{\sin^3 \psi / 2} \sin \psi \, d\psi.$$

In order to solve this integral we insert

$$\sin \psi/2 = z$$

which lead to the result

$$(6-5) \quad \Delta g(N_p) = N_p \cdot \frac{\gamma_e}{2a} \left[\frac{1}{\sin \psi_0/2} - 1 \right].$$

The following table gives an impression about the magnitude of this correction term for various cap sizes ψ_0 .

Table One. (N_p in meters, Δg in milligal)

ψ_0	$\Delta g(N_p)$
2°	$4.3 N_p$
5°	$1.7 N_p$
10°	$0.8 N_p$
20°	$0.4 N_p$

N_p [meter]geoidal undulation at point P
 $\Delta g(N_p)$ [mgal]error due to the neglecting of the
constant N_p

Because this correction term may become fairly large it must always be taken into account in the course of excluding the outer zones from the integration. After this we will compute the magnitude of the neglected term Δg_{oz} given by

$$\Delta g_{oz} = - \frac{\gamma_e}{4 \pi a} \int_{\psi=\psi_0}^{\pi} \int_{\alpha=0}^{2\pi} M(\psi) \cdot N \cdot \sin \psi \, d\psi \, d\alpha.$$

We introduce the discontinuous function

$$(6-6) \quad \overline{M}(\psi) = \begin{cases} 0 & \text{if } 0 \leq \psi < \psi_0 \\ M(\psi) & \text{if } \psi_0 \leq \psi \leq \pi \end{cases}$$

getting

$$(6-7) \quad \Delta g_{oz} = - \frac{\gamma_e}{4\pi a} \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} N \cdot \overline{M}(\psi) \sin\psi \, d\psi \, d\alpha.$$

The function $\overline{M}(\psi)$ can be developed into a series of Legendre polynomials

$$(6-8) \quad \overline{M}(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} U_n P_n(\cos\psi).$$

For the moment we will assume the coefficients U_n are known. A formula for the computation of the U_n will be derived afterwards. Inserting the series (6-8) into the integral (6-7) and interchange the order of summation and integration we get

$$(6-9) \quad \Delta g_{oz} = - \frac{\gamma_e}{8\pi a} \sum_{n=0}^{\infty} (2n+1) U_n \int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} N \cdot P_n(\cos\psi) \sin\psi \, d\psi \, d\alpha.$$

We can write N as a series of spherical harmonics

$$N = \sum_{n=0}^{\infty} N_n(\theta, \lambda).$$

As a solution of the double integral we obtain

$$\int_{\psi=0}^{\pi} \int_{\alpha=0}^{2\pi} N \cdot P_n(\cos\psi) \sin\psi \, d\psi \, d\alpha = \frac{4\pi N_n(\theta, \lambda)}{(2n+1)}$$

and with this intermediate result

$$(6-10) \quad \Delta g_{oz} = - \frac{\gamma_o}{2a} \sum_{n=2}^{\infty} U_n N_n(\theta, \lambda).$$

Expression (6-10) may be considered as the error at a certain point $P = P(\theta, \lambda)$, caused by neglecting the geoidal undulations beyond a circle of radius ψ_0 around this point P . However, what we want is not an individual error but its variance

$$(6-11) \quad \|\Delta g_{oz}\|^2 = \{\Delta g_{oz}^2\} = \frac{\gamma_o^2}{4a^2} \left\{ \left(\sum_{n=2}^{\infty} U_n N_n(\theta, \lambda) \right)^2 \right\}.$$

Starting with this expression we get, similar as in (Heiskanen-Moritz, 1967, p.261), after some manipulations

$$\|\Delta g_{oz}\|^2 = \frac{\gamma_o^2}{4a^2} \sum_{n=2}^{\infty} \sum_{n'=2}^{\infty} U_n U_{n'} \{N_n N_{n'}\}..$$

It is well-known that all $\{N_n N_{n'}\}$ are zero except for the case $n' = n$. Therefore we obtain finally

$$(6-12) \quad \|\Delta g_{oz}\|^2 = \frac{\gamma_o^2}{4a^2} \sum_{n=2}^{\infty} U_n^2 K_n.$$

By K_n we have denoted the degree variance coefficients of the disturbing potential respectively of the geoidal undulations itself.

It remains the evaluation of a suited expression for the coefficients U_n . We start with the integral expression

$$U_n = \int_{\psi_0}^{\pi} M(\psi) P_n(\cos\psi) \sin\psi \, d\psi$$

or

$$(6-13) \quad U_n = \frac{1}{4} \int_{\psi_0}^{\pi} \frac{1}{\sin^3 \psi/2} P_n(\cos \psi) \sin \psi \, d\psi.$$

The main difference to the similar problem of the computation of Molodenskii's coefficients Q_n (Heiskanen-Moritz, 1967, p. 260) consists in the fact, that the kernel $M(\psi)$ is much easier than Stokes' function. Introducing the new variable

$$z = \sin \psi/2, \quad t = \sin \psi_0/2$$

$$\cos \psi = (1 - 2z^2), \quad \sin \psi \, d\psi = 4z \, dz$$

we obtain,

$$(6-14) \quad U_n = - \int_1^t P_n(1 - z^2) \frac{dz}{z^3}.$$

We will develop the Legendre polynomials into a power series of z . For this purpose we start with the well-known series representation of the Legendre polynomials

$$(6-15) \quad P_n(1 - 2z^2) = 2^{-n} \sum_{k=0}^r (-1)^k \frac{(2n - 2k)!}{k! (n - k)! (n - 2k)!} (1 - 2z^2)^{n-2k},$$

$$r = \text{INT}(n + 2).$$

Because of the binomial expansion

$$(6-16) \quad (a-b)^n = \sum_{\nu=0}^n (-1)^\nu \binom{n}{\nu} a^{(n-\nu)} b^\nu$$

we get

$$(1-2z^2)^{n-2k} = \sum_{\nu=0}^{n-2k} (-1)^\nu \frac{(n-2k)! 2^\nu}{\nu! (n-2k-\nu)!} z^{2\nu}$$

or

$$(6-17) \quad \frac{(1-2z^2)^{n-2k}}{z^2} = \sum_{\nu=0}^{n-2k} (-1)^\nu \frac{(n-2k)! 2^\nu}{\nu! (n-2k-\nu)!} z^{2(\nu-1)}$$

With this transformation of Legendre polynomials we can write the integral (6-14) in the following form

$$U_n = - \int_1^t \sum_{k=0}^r A_{nk} \sum_{\nu=0}^{n-2k} B_{nk\nu} \cdot z^{2(\nu-1)}$$

where

$$A_{nk} = (-1)^k \frac{(2n-2k)! 2^{-n}}{k! (n-k)! (n-2k)!}$$

and

$$B_{nk\nu} = (-1)^\nu \frac{(n-2k)! 2^\nu}{\nu! (n-2k-\nu)!}$$

We interchange the order of integration and summation and because of $2(\nu-1) \neq -1$ we can perform the integration with the result

$$(6-18) \quad U_n = - \sum_{k=0}^r A_{nk} \sum_{\nu=0}^{n-2k} B_{nk\nu} \frac{1}{2\nu-1} [t^{(2\nu-1)} - 1].$$

We will give this power series a more convenient form. First we can write

$$U_n = - \sum_{k=0}^r \sum_{\nu=0}^{n-2k} (-1)^{\nu+k} \frac{1}{2^{(n-\nu)}} \frac{(2n-2k)!}{\nu! k! (n-k)! (n-2k-\nu)!} \cdot \frac{1}{2\nu-1} [t^{(2\nu-1)} - 1].$$

Here we have of course $r = \text{INT}(n/2)$. Now we will change the order of summation, that is, we will sum first over ν . Because of the case $k=0$, the summation over ν has the range from 0 to n . Because of $\nu = n - 2k$ we have for a summation over k a range from $k=0$ to $k=p$ with $p = \text{INT}((n-\nu)/2)$. We arrive therefore at the desired result with the expression

$$(6-19) \quad U_n = - \sum_{\nu=0}^n \sum_{k=0}^p (-1)^{\nu+k} \frac{1}{2^{(n-\nu)}} \frac{(2n-2k)!}{\nu! k! (n-k)! (n-2k-\nu)!} \cdot \frac{1}{2\nu-1} [t^{(2\nu-1)} - 1]$$

with

$$p = \text{INT}((n-\nu)/2) \\ t = \sin\psi_0/2.$$

The final result (6-19) was checked by an analytical integration of formula (6-14) up to an order of $n=3$.

For an estimate of the error due to the neglecting of the outer zones we use the degree variances from the covariance function (1-8), computed by Tscherning and Rapp

$$K_n = \left(\frac{a^2}{\gamma_e^2} \right) \frac{425.28}{(n-1)(n-2)(n+24)} s^{n+1}$$

with

$$s = 0.999617.$$

In this case we get as error

$$(6-20) \quad m^2(\Delta g_{oz}) = \frac{425.28}{4} \sum_{n=3}^{\infty} \frac{s^{n+1}}{(n-1)(n-2)(n+24)} U_n^2 [\text{mgal}]^2.$$

A computer program was written in order to get numerical results. These results are given in the following table for various radius ψ_0 of the outer zone and different degree n_0 of a reference field. The dimension of $m(\Delta g_{oz})$ is milligal.

Table Two. Influence of the Zone Beyond the Radius ψ_0 for Reference Model of Degree n_0 .

$\psi_0 \backslash n_0$	2	8	16
2°	10.2	0.5	0.1
5°	3.3	0.1	0.0
10°	1.1	0.0	0.0
20°	0.2	0.0	0.0

If we know exactly a reference field up to a degree $n_0 = 16$, then it seems generally sufficient to carry out the integration up to a circle of $\psi_0 = 2^\circ$. So it may be also possible to compute gravity anomalies near the coast of the continents from the altimeter data.

7. Summary

A detailed description of a model for the recovery of gravity anomalies from altimeter data with 10 cm accuracy is given in Section Two. The disturbing potential at the sphere with radius a is computed from the geoidal undulations at the earth's surface. After this, gravity anomalies at this sphere are recovered and then these anomalies reduced to values at the earth's surface. The practical solution consists of small correction terms to the inverse Stokes' formula.

The influence of the sea surface topography is discussed especially in the case of meandering currents such as the Gulf Stream. From the influence of the unknown meandering of these currents a rough error estimate indicates an expected error of

$$\pm 3 \text{ mgal} \leq m(\Delta g_{ss}) \leq \pm 7 \text{ mgal}.$$

$m(\Delta g_{ss})$ error of the gravity anomaly due to
sea surface topography

The most critical error influence is probably due to the data errors and due to the neglect of the inner zone. It was assumed that mean geoidal undulations are given all over the earth in blocks of size b and with an accuracy of $m(N)$.

Point errors at the midpoint of the blocks can be estimated with

$$m^2(\Delta g_m) = \frac{6.1^2 m^2(N)}{b^2} + 0.014^2 \left\| \frac{\partial \Delta g}{\partial r} \right\|^2 \cdot b^2$$

$m(\Delta g_m)$ error of the gravity anomaly in [mgal]

$m(N)$ error of the mean geoidal undulations in [cm]

b block size of the mean geoidal undulations in [km]

$\left\| \frac{\partial \Delta g}{\partial r} \right\|$ mean value of the vertical gravity gradient
in Eötvös units

For $m(N) = \pm 10$ cm, $b \doteq 13$ km and $\left\| \frac{\partial \Delta g}{\partial r} \right\| \doteq 24$ EU we get an estimate of

$$m(\Delta g_m) = \pm 6 \text{ mgal.}$$

An estimate of the accuracy of mean anomalies of block size c ,

$$c > b,$$

gives under certain natural assumptions

$$\frac{b}{c} m(\Delta g_m) \leq m_c(\Delta g_m) \leq m(\Delta g_m).$$

$m_c(\Delta g_m)$ error of the mean anomaly of
block size c .

With the numbers above and a block size $c = 1^\circ$ we obtain

$$\pm 1 \text{ mgal} \leq m_c(\Delta g_m) \leq \pm 6 \text{ mgal.}$$

The results are in good agreement with the results of the error considerations in (Rapp, 1974), which are based on least squares collocation.

In the last section the error of neglecting an outer zone in the inverse Stokes' integral is investigated. Such a procedure may be necessary because it is not yet pos-

sible to compute geoidal undulations from altimeter data on land. If the harmonics of lower order up to a degree $n_0 = 16$ are known, the range of integration can be restricted to a few degrees without a significant additional error. However, a certain correction has to be taken into account in this case.

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